WITTEN DEFORMATION OF RAY-SINGER ANALYTIC TORSION

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ABSTRACT. Let F be a flat vector bundle over a compact Riemannian manifold M and let $f : M \to \mathbb{R}$ be a self-indexing Morse function. Let g^F be a smooth Euclidean metric on F, let $g_t^F = e^{-2tf}g^F$ and let $\rho^{RS}(t)$ be the Ray-Singer analytic torsion of F associated to the metric g_t^F. Assuming that ∇f satisfies the Morse-Smale transversality conditions, we provide an asymptotic expansion for $\log \rho^{RS}(t)$ for $t \to \infty$ of the form $a_0 + a_1 t + b \log\left(\frac{t}{\pi}\right) + o(1)$. We present explicit formulae for coefficients a_0, a_1 and b. In particular, we show that b is a half integer.

0. INTRODUCTION

0.1. The Ray-Singer analytic torsion. Let M be a compact manifold of dimension n and let F be a flat vector bundle on M. Let g^F and g^{TM} be smooth metrics on F and TM respectively.

In [RS] Ray and Singer introduced a numerical invariant of these data which is called the *Ray-Singer analytic torsion* of F and which we shall denote by ρ^{RS}.

0.2. The Witten deformation. Suppose $f : M \to \mathbb{R}$ is a Morse function. For $t > 0$, we denote by g_t^F the smooth metric on F

$$(0.1) \qquad g_t^F = e^{-2tf}g^F.$$

Let $\rho^{RS}(t)$ be the Ray-Singer torsion on F associated to the metrics g_t^F and g^{TM}.

Denote by ∇f the gradient vector field of f with respect to the metric g^{TM}. Let B be the finite set of zeroes of ∇f.

We shall assume that the following conditions are satisfied (cf. [BFK3, page 5]):

(1) $f : M \to \mathbb{R}$ is a self-indexing Morse function (i.e. $f(x) = index(x)$ for any critical point x of f).

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(2) The gradient vector field ∇f satisfies the Smale transversality conditions [Sm1, Sm2] (for any two critical points x and y of f the stable manifold $W^s(x)$ and the unstable manifold $W^u(y)$, with respect to ∇f, intersect transversally).

(3) For any $x \in B$, the metric g^F is flat near B and there is a system of coordinates $y = (y^1, \ldots, y^n)$ centered at x such that near x

$$(0.2) \qquad g^{TM} = \sum_{i=1}^{n} |dy^i|^2, \quad f(y) = f(x) - \frac{1}{2} \sum_{i=1}^{index(x)} |y^i|^2 + \frac{1}{2} \sum_{i=index(x)+1}^{n} |y^i|^2.$$

0.3. Asymptotic expansion of the torsion. Burghelea, Friedlander and Kappeler ([BFK3]) have shown that the function $\log \rho^{RS}(t)$ has asymptotic expansion for $t \to \infty$ of the form

$$(0.3) \qquad \log \rho^{RS}(t) = \sum_{j=0}^{n+1} a_j t^j + b \log t + o(1).$$

The coefficient a_0 is calculated in [BFK3] in terms of the parametrix of the Laplace-Beltrami operator.

In the present paper we shall calculate all coefficients in the asymptotic expansion [BFK3]. In fact, we shall show that the coefficients $a_j = 0$ for $j > 1$ and the coefficient b is a half integer.

0.4. To formulate our result, we need to introduce some notation (cf. [BZ1]).

Let ∇^{TM} be the Levi-Civita connection on TM corresponding to the metric g^{TM}, and let $e(TM, \nabla^{TM})$ be the associated representative of the Euler class of TM in Chern-Weil theory.

Let $\psi(TM, \nabla^{TM})$ be the Mathai-Quillen ([MQ]) $n-1$ current on TM (see also [BGS, Section 3] and [BZ1, Section IIId]).

Let ∇^F be the flat connection on F and let $\theta(F, g^F)$ be the 1-form on M defined by (cf. [BZ1, Section IVd])

$$(0.4) \qquad \theta(F, g^F) = \text{Tr}\left[(g^F)^{-1}\nabla^F g^F\right].$$

Set

$$(0.5) \qquad \begin{aligned} \chi(F) &= \sum_{i=0}^{n}(-1)^i \dim H^i(M, F), \\ \chi'(F) &= \sum_{i=0}^{n}(-1)^i i \dim H^i(M, F). \end{aligned}$$

Let $\rho^{\mathcal{M}}$ be the torsion of the Thom-Smale complex (cf. Section 1).

Theorem 0.5. *The function* $\log \rho^{RS}(t)$ *admits an asymptotic expansion for* $t \to \infty$ *of the form*

$$(0.6) \qquad \log \rho^{RS}(t) = a_0 + a_1 t + b \log\left(\frac{t}{\pi}\right) + o(1),$$

where the coefficients a_0, a_1 *and* b *are given by the formulas*

$$(0.7) \qquad a_0 = \log \rho^{\mathcal{M}} - \frac{1}{2}\int_M \theta(F, g^F)(\nabla f)^* \psi(TM, \nabla^{TM});$$

$$(0.8) \qquad a_1 = -\operatorname{rk}(F)\int_M f e(TM, \nabla^{TM}) + \chi'(F);$$

and

$$(0.9) \qquad b = \frac{n}{4}\chi(F) - \frac{1}{2}\chi'(F).$$

Remark 0.6. Note that $\chi(F) = 0$ if n is odd. Hence, (0.9) implies that the coefficient b is a half integer for any n.

0.7. The method of the proof. Our method is completely different from that of [BFK3]. In [BFK3] the asymptotic expansion (0.3) is proved by direct analytic arguments and, then is applied to get a new proof of the Ray-Singer conjecture [RS].

In the present paper we use the Bismut-Zhang extension of this conjecture ([BZ1]) in order to obtain the Theorem 0.5.

1. MILNOR METRIC AND MILNOR TORSION

In this section we follow [BZ1, Chapter I].

1.1. The determinant line of the cohomology. Let $H^\bullet(M, F) = \bigoplus_{i=0}^n H^i(M, F)$ be the cohomology of M with coefficients in F and let $\det H^\bullet(M, F)$ be the line

$$(1.1) \qquad \det H^\bullet(M, F) = \bigotimes_{i=0}^n \left(\det H^i(M, F)\right)^{(-1)^i}.$$

1.2. The Thom-Smale complex. Suppose $f : M \to \mathbb{R}$ is a Morse function satisfying the Smale transversality conditions [Sm1, Sm2] (for any two critical points x and y of f the stable manifold $W^s(x)$ and the unstable manifold $W^u(y)$, with respect to ∇f, intersect transversally).

Let B be the set of critical points of f. If $x \in B$, we denote by F_x the fiber of F over x and by $[W^u(x)]$ the real line generated by $W^u(x)$. For $0 \le i \le n$, set

$$(1.2) \qquad\qquad C^i(W^u, F) = \bigoplus_{\substack{x \in B \\ index(x)=i}} [W^u(x)]^* \otimes_{\mathbb{R}} F_x.$$

By a basic result of Thom ([Th]) and Smale ([Sm2]) (see also [BZ1, pages 28–30]), there is a well defined linear operators

$$\partial : C^i(W^u, F) \to C^{i+1}(W^u, F),$$

such that the pair $(C^\bullet(W^u, F), \partial)$ is a complex and there is a canonical identification of $\mathbb{Z}$-graded vector spaces $H^\bullet(C^\bullet(W^u, F), \partial) \simeq H^\bullet(M, F)$. By [KMu] there is a canonical isomorphism

$$(1.3) \qquad\qquad \det H^\bullet(M, F) \simeq \det C^\bullet(W^u, F).$$

1.3. The Milnor metric. The metric g^F on F determines the structure of Euclidean vector space on $C^\bullet(W^u, F)$.

Definition 1.4. The *Milnor metric* $\| \cdot \|^{\mathcal{M}}_{\det H^\bullet(M,F)}$ on the line $\det H^\bullet(M, F)$ is the metric corresponding to the obvious metric on $\det C^\bullet(W^u, F)$ via the canonical isomorphism (1.3).

Remark 1.5. By Milnor [Mi1, Theorem 9.3], if g^F is a flat metric on F then the Milnor metric coincides with the Reidemeister metric defined through a smooth triangulation of M. In this case $\| \cdot \|^{\mathcal{M}}_{\det H^\bullet(M,F)}$ does not depend upon F and g^{TM} and, hence, is a topological invariant of the flat Euclidean vector bundle F.

1.6. The Milnor torsion. Let ∂^* be the adjoint of ∂ with respect to the Euclidean structure on $C^\bullet(W^u, F)$. Using the finite dimensional Hodge theory, we have the canonical identification

$$(1.4) \quad H^i(C^\bullet(W^u, F), \partial) \simeq \{v \in C^i(W^u, F) : \ \partial v = 0, \partial^* v = 0\}, \quad 0 \le i \le n.$$

As a vector subspace of $C^i(W^u, F)$, the vector space in the right-hand side of (1.4) inherits the Euclidean metric. We denote by $| \cdot |^{\mathcal{M}}_{\det H^\bullet(M,F)}$ the corresponding metric on $\det H^\bullet(M, F)$.

The metrics $\| \cdot \|^{\mathcal{M}}_{\det H^\bullet(M,F)}$ and $| \cdot |^{\mathcal{M}}_{\det H^\bullet(M,F)}$ do not coincide in general. We shall describe the discrepancy.

Set $\Delta = \partial \partial^* + \partial^* \partial$ and let $P : C^\bullet(W^u, F) \to \operatorname{Ker} \Delta$ be the orthogonal projection. Set $\Pi^\perp = 1 - \Pi$.

Let N and τ be the operators on $C^\bullet(W^u, F)$ acting on $C^i(W^u, F)$ $(0 \le i \le n)$ by multiplication by i and $(-1)^i$ respectively. If $A \in \mathrm{End}\,(C^\bullet(W^u, F))$, we define the supertrace $\mathrm{Tr}_s[A]$ by the formula

$$(1.5) \qquad \mathrm{Tr}_s[A] = \mathrm{Tr}[\tau A].$$

For $s \in \mathbb{C}$, set

$$\eta^{\mathcal{M}}(s) = -\mathrm{Tr}_s\left[N(\Delta)^{-s}\Pi^\perp\right].$$

Definition 1.7. The *Milnor torsion* is the number

$$(1.6) \qquad \rho^{\mathcal{M}} = \exp\left(\frac{1}{2}\frac{d\eta^{\mathcal{M}}(0)}{ds}\right).$$

The following result is proved in [BGS, Proposition 1.5]

$$(1.7) \qquad \|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)} = |\cdot|^{\mathcal{M}}_{\det H^\bullet(M,F)} \cdot \rho^{\mathcal{M}}.$$

1.8. Deformation of Milnor metric. The metric $\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}$ depends on the metric g^F. Let $g^F_t = e^{-2tf}g^F$ and let $\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F),t}$ be the corresponding Milnor metric. Set

$$(1.8) \qquad \widetilde{\chi}'(F) = \mathrm{rk}(F)\sum_{x \in B}(-1)^{index(x)}index(x).$$

As f is a self-indexing Morse function

$$\mathrm{rk}(F)\sum_{x \in B}(-1)^{index(x)}f(x) = \widetilde{\chi}'(F).$$

Obviously,

$$(1.9) \qquad \|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F),t} = e^{-t\widetilde{\chi}'(F)} \cdot \|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}.$$

2. Ray-Singer metric and Ray-Singer torsion

2.1. L_2 metric on the determinant line. Let $(\Omega^\bullet(M,F), d^F)$ be the de Rahm complex of the smooth sections of $\bigwedge(T^*M) \otimes F$ equipped with the coboundary operator d^F. The cohomology of this complex is canonically isomorphic to $H^\bullet(M,F)$.

Let $*$ be the Hodge operator associated to the metric g^{TM}. We equip $\Omega^\bullet(M,F)$ with the inner product

$$(2.1) \qquad \langle \alpha, \alpha' \rangle_{\Omega^\bullet(M,F)} = \int_M \langle \alpha \wedge *\alpha' \rangle_{g^F}.$$

By Hodge theory, we can identify $H^\bullet(M,F)$ to the corresponding harmonic forms in $\Omega^\bullet(M,F)$. These forms inherit the Euclidean product (2.1). Thus the line $\det H^\bullet(M,F)$ inherits a metric $|\cdot|^{RS}_{\det H^\bullet(M,F)}$, which is also called the L_2 metric.

2.2. The Ray-Singer torsion. Let d^{F*} be the formal adjoint of d^F with respect to the metrics g^{TM} and g^F.

Set $\Delta = d^F d^{F*} + d^{F*} d^F$ and let $\Pi : \Omega^\bullet(M, F) \to \mathrm{Ker}\,\Delta$ be the orthogonal projection. Set $\Pi^\perp = 1 - \Pi$.

Let N be the operator defining the $\mathbb{Z}$-grading of $\Omega^\bullet(M, F)$, i.e. N acts on $\Omega^i(M, F)$ by multiplication by i.

If an operator $A : \Omega^\bullet(M, F) \to \Omega^\bullet(M, F)$ is trace class, we define its supertrace $\mathrm{Tr}_s[A]$ as in (1.5).

For $s \in \mathbb{C}$, set

$$\eta^{RS}(s) = -\,\mathrm{Tr}_s\left[N(\Delta)^{-s}\Pi^\perp\right].$$

By a result of Seeley [Se], $\eta^{RS}(s)$ extends to a meromorphic function of $s \in \mathbb{C}$, which is holomorphic at $s = 0$.

Definition 2.3. The *Ray-Singer torsion* is the number

$$(2.2) \qquad \rho^{RS} = \exp\left(\frac{1}{2}\frac{d\eta^{RS}(0)}{ds}\right).$$

2.4. The Ray-Singer metric. We now remind the following definition (cf. [BZ1, Definition 2.2]):

Definition 2.5. The *Ray-Singer metric* $\|\cdot\|^{RS}_{\det H^\bullet(M,F)}$ on the line $\det H^\bullet(M, F)$ is the product

$$(2.3) \qquad \|\cdot\|^{RS}_{\det H^\bullet(M,F)} = |\cdot|^{RS}_{\det H^\bullet(M,F)} \cdot \rho^{RS}.$$

Remark 2.6. When M is odd dimensional, Ray and Singer [RS, Theorem 2.1] proved that the metric $\|\cdot\|^{RS}_{\det H^\bullet(M,F)}$ is a topological invariant, i.e. does not depend on the metrics g^{TM} or g^F. Bismut and Zhang [BZ1, Theorem 0.1] described explicitly the dependents of $\|\cdot\|^{RS}_{\det H^\bullet(M,F)}$ on g^{TM} and g^F in the case when $\dim M$ is odd.

2.7. Bismut-Zhang theorem. Let ∇^{TM} be the Levi-Civita connection on TM corresponding to the metric g^{TM}, and let $e(TM, \nabla^{TM})$ be the associated representative of the Euler class of TM in Chern-Weil theory.

Let $\psi(TM, \nabla^{TM})$ be the Mathai-Quillen ([MQ]) $n - 1$ current on TM (see also [BGS, Section3] and [BZ1, Section IIId]).

Let ∇^F be the flat connection on F and let $\theta(F, g^F)$ be the 1-form on M defined by (cf. [BZ1, Section IVd])

$$(2.4) \qquad \theta(F, g^F) = \mathrm{Tr}\left[(g^F)^{-1}\nabla^F g^F\right].$$

Now we remind the following theorem by Bismut and Zhang [BZ1, Theorem 0.2].

Theorem 2.8 (Bismut-Zhang). *The following identity holds*

$$(2.5) \qquad \log\left(\frac{\|\cdot\|^{RS}_{\det H^\bullet(M,F)}}{\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}}\right)^2 = -\int_M \theta(F,g^F)(\nabla f)^*\psi(TM,\nabla^{TM}).$$

2.9. Dependence on the metric. The metrics $\|\cdot\|^{RS}_{\det H^\bullet(M,F)}$ and $\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}$ depend, in general, on the metric g^F. Let $g^F_t = e^{-2tf}g^F$ and let $\|\cdot\|^{RS}_{\det H^\bullet(M,F),t}$ and $\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F),t}$ be the Ray-Singer and Milnor metrics on $\det H^\bullet(M,F)$ associated to the metrics g^F_t and g^{TM}.

By [BZ1, Theorem 6.3]

$$(2.6) \quad \int_M \theta(F,g^F_t)(\nabla f)^*\psi(TM,\nabla^{TM}) =$$

$$\int_M \theta(F,g^F)(\nabla f)^*\psi(TM,\nabla^{TM}) + 2t\operatorname{rk}(F)\int_M fe(TM,\nabla^{TM}) - 2t\widetilde{\chi}'(F).$$

From (1.9), (2.5) and (2.6), we get

$$(2.7) \quad \log\left(\frac{\|\cdot\|^{RS}_{\det H^\bullet(M,F),t}}{\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}}\right)^2 =$$

$$-\int_M \theta(F,g^F)(\nabla f)^*\psi(TM,\nabla^{TM}) - 2t\operatorname{rk}(F)\int_M fe(TM,\nabla^{TM}).$$

3. The main result

In this section we prove Theorem 0.5, which we restate for convenience.

Theorem 3.1. *The function $\log\rho^{RS}(t)$ admits an asymptotic expansion for $t\to\infty$ of the form*

$$(3.1) \qquad \log\rho^{RS}(t) = a_0 + a_1 t + b\log\left(\frac{t}{\pi}\right) + o(1),$$

where the coefficients a_0, a_1 and b are given by the formulas

$$(3.2) \qquad a_0 = \log\rho^{\mathcal{M}} - \frac{1}{2}\int_M \theta(F,g^F)(\nabla f)^*\psi(TM,\nabla^{TM});$$

$$(3.3) \qquad a_1 = -\operatorname{rk}(F)\int_M fe(TM,\nabla^{TM}) + \chi'(F);$$

and

$$(3.4) \qquad b = \frac{n}{4}\chi(F) - \frac{1}{2}\chi'(F).$$

Proof. For each $t > 0$ we equip $\Omega^\bullet(M, F)$ with the inner product

$$(3.5) \qquad \langle \alpha, \alpha' \rangle_{\Omega^\bullet(M,F),t} = \int_M \langle \alpha \wedge *\alpha' \rangle_{g_t^F}.$$

and we denote by $|\cdot|_{\det H^\bullet(M,F),t}^{RS}$ the L_2 metric on $\det H^\bullet(M, F)$ (cf. Section 2.1) associated to this inner product.

From (1.7), (2.3) and (2.7), we get

$$(3.6) \quad \log \rho^{RS}(t) = -\frac{1}{2} \int_M \theta(F, g^F)(\nabla f)^* \psi(TM, \nabla^{TM}) -$$

$$t \operatorname{rk}(F) \int_M f e(TM, \nabla^{TM}) + \log \rho^{\mathcal{M}} + \log \left(\frac{|\cdot|_{\det H^\bullet(M,F)}^{\mathcal{M}}}{|\cdot|_{\det H^\bullet(M,F),t}^{RS}} \right).$$

Let d_t^{F*} be the formal adjoint of d^F with respect to the inner product (3.5). Set $\Delta_t = d^F d_t^{F*} + d_t^{F*} d^F$.

Let $\Omega_t^{\bullet,[0,1]}(M, F)$ be the direct sum of the eigenspaces of Δ_t associated to eigenvalues $\lambda \in [0, 1]$. The pair $(\Omega_t^{\bullet,[0,1]}(M, F), d^F)$ is a subcomplex of $(\Omega^\bullet(M, F), d^F)$.

We denote by $\|\cdot\|_{\Omega^\bullet(M,F)}$ the norm on $\Omega^\bullet(M, F)$ determined by inner product (3.5), and by $\|\cdot\|_{C^\bullet(W^u,F)}$ the norm on $C^\bullet(W^u, F)$ determined by g^F (cf. Section 1.3).

In the sequel, $o(1)$ denotes an element of $\operatorname{End}\left(C^\bullet(W^u, F) \right)$ which preserves the $\mathbb{Z}$-grading and is $o(1)$ as $t \to \infty$.

It is shown in [BZ2, Theorem 6.9] that if $t > 0$ is large enough, there exists an isomorphism

$$e_t : C^\bullet(W^u, F) \to \Omega_t^{\bullet,[0,1]}(M, F)$$

of $\mathbb{Z}$-graded Euclidean vector spaces such that

$$(3.7) \qquad\qquad\qquad e_t^* e_t = 1 + o(1).$$

By [BZ2, Theorem 6.11], for any $t > 0$ there is a quasi-isomorphism of complexes

$$P_t : \left(\Omega_t^{\bullet,[0,1]}(M, F), d^F \right) \to \left(C^\bullet(W^u, F), \partial \right),$$

which induces the canonical isomorphism

$$(3.8) \qquad H^\bullet(M, F) \simeq H^\bullet(\Omega_t^{\bullet,[0,1]}(M, F), d^F) \simeq H^\bullet(C^\bullet(W^u, F), \partial)$$

and such that

$$(3.9) \qquad\qquad\qquad P_t e_t = e^{tN} \left(\frac{t}{\pi} \right)^{n/4 - N/2} \left(1 + o(1) \right).$$

Here $e^{tN} \left(\frac{t}{\pi} \right)^{n/4 - N/2}$ denotes the operator on $C^\bullet(W^u, F)$ acting on $C^i(W^u, F)$ by multiplication by $e^{ti} \left(\frac{t}{\pi} \right)^{n/4 - i/2}$.

It follows from (3.9), that, for $t > 0$ large enough, P_t is one to one.

From (3.7), (3.9) we get

$$(3.10) \qquad P_t P_t^* = e^{2tN} \left(\frac{t}{\pi} \right)^{n/2-N} \left(1 + o(1) \right).$$

Fix $\sigma \in H^i(M, F)$ $(0 \le i \le n)$ and let $\omega_t \in \mathrm{Ker}\,\Delta_t$ be the harmonic form representing σ.

Let $\Pi : C^\bullet(W^u, F) \to \mathrm{Ker}(\partial\partial^* + \partial^*\partial)$ be the orthogonal projection. Then $\Pi P_t \omega_t \in C^i(W^u, F)$ corresponds to σ via the canonical isomorphisms (1.4), (3.8).

Obviously,

$$(3.11) \qquad P_t \omega_t \in \mathrm{Ker}\,\partial, \qquad e^{2ti} \left(\frac{t}{\pi} \right)^{n/2-i} \left(P_t^* \right)^{-1} \omega_t \in \mathrm{Ker}\,\partial^*.$$

By (3.10), we get $e^{2ti} \left(\frac{t}{\pi} \right)^{n/2-i} \left(P_t^* \right)^{-1} \omega_t = \left(1 + o(1) \right) P_t \omega_t$. Then (3.11) implies

$$(3.12) \qquad \left\| \Pi P_t \omega_t \right\|_{C^\bullet(W^u, F)} = \left\| P_t \omega_t \right\|_{C^\bullet(W^u, F)} \left(1 + o(1) \right).$$

From (3.10), (3.12) we obtain

$$(3.13) \qquad \left\| \Pi P_t \omega_t \right\|_{C^\bullet(W^u, F)} = e^{ti} \left(\frac{t}{\pi} \right)^{n/4-i/2} \left\| \omega_t \right\|_{\Omega^\bullet(M, F), t} \left(1 + o(t) \right).$$

It follows from (3.13) and from the definitions of the metrics $|\cdot|^{\mathcal{M}}_{\det H^\bullet(M, F)}$, $|\cdot|^{RS}_{\det H^\bullet(M, F), t}$ that

$$(3.14) \quad \log \left(\frac{|\cdot|^{\mathcal{M}}_{\det H^\bullet(M, F)}}{|\cdot|^{RS}_{\det H^\bullet(M, F), t}} \right) = t\chi'(F) + \left(\frac{n}{4}\chi(F) - \frac{1}{2}\chi'(F) \right) \log \left(\frac{t}{\pi} \right) + o(1).$$

From (3.6), (3.14) we get

$$(3.15) \quad \log \rho^{RS}(t) =$$
$$-\frac{1}{2} \int_M \theta(F, g^F)(\nabla f)^* \psi(TM, \nabla^{TM}) - t\,\mathrm{rk}(F) \int_M f e(TM, \nabla^{TM}) +$$
$$\log \rho^{\mathcal{M}} + t\chi'(F) + \left(\frac{n}{4}\chi(F) - \frac{1}{2}\chi'(F) \right) \log \left(\frac{t}{\pi} \right) + o(1).$$

The proof of Theorem 3.1 is completed. $\quad \square$

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